

Color superconductivity and nondecoupling phenomena in $(2+1)$ -dimensional QCD

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The possibility of generating a color superconducting state in $(2+1)$ -dimensional QCD is analyzed. The gap equation in the leading, hard-dense loop improved, one-gluon exchange approximation is derived and solved. The magnitude of the order parameter is proportional to a power of the coupling constant. For an asymptotically large chemical potential, a qualitatively new [with respect to the $(3+1)$ -dimensional case] phenomenon of nondecoupling of the fermion pairing dynamics from the infrared one is revealed and discussed.

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I. INTRODUCTION

The study of color superconductivity has attracted a great deal of attention in the past few years. The first theoretical studies of the subject appeared over twenty years ago [1,2]. For many reasons, however, its importance was not fully appreciated until recently [3,4]. The last couple of years were marked by intensive theoretical studies, producing further understanding of the subtleties of physics behind color superconductivity [5–27] (see Refs. [28,29] for a review).

The color superconducting phase is expected to appear in sufficiently dense quark matter. It is quite likely, therefore, that it exists inside some neutron stars [30]. Of course, the hope is that the superconducting order parameter is large enough to result in clear observational signatures. While this still remains to be confirmed, more theoretical work is needed before any solid quantitative predictions would be available.

In this paper, we study the color superconducting phase in $(2+1)$ -dimensional QCD at zero temperature and finite density. This theory is superrenormalizable and, therefore, asymptotically free. The conventional wisdom is that at asymptotically high quark density the dynamics in asymptotically free theories is weakly interacting and therefore drastically simplified. This is the case in $(3+1)$ -dimensional QCD [6–9]. One of our goals is to check the validity of this picture in $(2+1)$ dimensions. We study the problem of color superconductivity using the conventional method of the Schwinger-Dyson (SD) equation in the hard-dense loop (HDL) improved one-gluon exchange approximation. By solving the SD equation analytically, we obtain an approximate solution for the color superconducting gap. The result is proportional to a power of the coupling constant. This contrasts the situation in $(3+1)$ -dimensional QCD, where the expression for the gap is nonanalytic in the coupling constant [$\Delta \sim \mu \exp(-C/\sqrt{\alpha_s})$ with μ being the quark chemical potential and $\alpha_s \equiv \alpha_s(\mu)$] when the coupling α_s is weak [5–9].

Is the conventional wisdom about weakly interacting dynamics at high quark density in $(2+1)$ -dimensional QCD justified? A detailed analysis of the solution of the SD equation in this theory reveals a qualitatively new [with respect to the $(3+1)$ -dimensional case] phenomenon: nondecoupling of the dynamics of the fermion pairing from the nonperturbative infrared one, even in the case of asymptotically large values of the chemical potential. We argue that this phenomenon is common in lower dimensional models. The relevance of the $1/N_f$ expansion, with N_f the number of fermion flavors, for solving difficulties induced by this phenomenon is pointed out.

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This paper is organized as follows. In Sec. II, we discuss different types of the Dirac and Majorana masses, paying special attention to their transformation properties under parity which itself could be defined in several ways. We argue that, in the QCD model at hand, only that color breaking Majorana mass is generated which preserves the global $SU(2)$ “pre-flavor” symmetry. The hard-dense loop gluon propagator is derived in Sec. III. Then, in Sec. IV, we derive the SD equation in the HDL improved rainbow approximation. We solve the equation analytically and present an approximate solution. In Sec. V the nondecoupling phenomenon taking place for large values of the chemical potential in $(2+1)$ -dimensional QCD is discussed. Our conclusions are given in Sec. VI. In Appendix A, we describe our notation and list some useful formulas, used throughout the main text. And in Appendix B, we give an approximate analytical solution of the gap equation.

II. ORDINARY AND MAJORANA MASS TERMS

The $(2+1)$ -dimensional models are somewhat different from their analogues in $(3+1)$ dimensions. In particular, the global “flavor” symmetries have somewhat different status. From studying color superconductivity in different models, we know that often color breaking is accompanied by locking with flavor symmetries [12,16]. Before studying the SD equation for a possibility of color superconducting gap, we discussed some general symmetry properties of all possible Dirac and Majorana masses in $(2+1)$ -dimensional QCD.

In $(2+1)$ dimensions, the irreducible spinor representation is two dimensional. For many purposes, however, it is more convenient to work with the four component spinors that contain two different irreducible representations. To make a connection with the usual terminology in $(3+1)$ dimensions, we could assign a notion of flavor to each four component spinor, while using “pre-flavor” for two component irreducible spinors.

In the case of four component spinors, the kinetic term of quarks,

$$\mathcal{L}_{kin} = \bar{\psi} (i\gamma^\mu D_\mu + \mu\gamma^0) \psi, \quad (1)$$

has $U(1)_V \times SU(2)$ (pre-flavor) symmetry [31–34], generated by

$$\mathcal{I}, \quad T^1 = \gamma^3\gamma^5, \quad T^2 = i\gamma^3, \quad T^3 = \gamma^5. \quad (2)$$

One could also add a Dirac mass term that respects this symmetry,

$$m_1 \bar{\psi} T^1 \psi. \quad (3)$$

This mass, however, is odd under parity

$$\psi(x_1, x_2) \rightarrow \gamma^1 \psi(-x_1, x_2), \quad (4)$$

and, furthermore, there are no modification of the parity transformation (there are at least three other possible definitions, $\psi \rightarrow \gamma^0\gamma^2\psi'$, $\psi \rightarrow \gamma^5\gamma^1\psi'$ and $\psi \rightarrow \gamma^3\gamma^1\psi'$, respectively) that would make the mass term in Eq. (3) parity even. Also, having the mass term as in Eq. (3), would lead to the generation of the Chern-Simons term at one loop order.

There are no other mass terms which would respect the $U(1)_V \times SU(2)$ (pre-flavor) symmetry [31–36]. For example, each of the following mass terms

$$m_0 \bar{\psi} \psi, \quad im_2 \bar{\psi} T^2 \psi, \quad im_3 \bar{\psi} T^3 \psi, \quad (5)$$

breaks the symmetry down to $U(1)_V \times U(1)$, generated by (\mathcal{I}, T^1) , (\mathcal{I}, T^3) , and (\mathcal{I}, T^2) , respectively. Note that the last two terms are even under parity defined in Eq. (4). Although no definition of parity exists that would be respected by all three mass terms in Eq. (5), a different definition of parity ($\psi \rightarrow \gamma^5\gamma^1\psi'$ or $\psi \rightarrow \gamma^3\gamma^1\psi'$) could make the first term in Eq. (5) parity even. Also, since the $SU(2)$ transformations could change each term in Eq. (5) into another, all three of them are physically equivalent.

As a result of spontaneous breaking of the color symmetry, different kinds of Majorana masses could be generated. In analogy with the ordinary masses, the following four structures are possible:

$$\Delta \bar{\psi}^C \psi, \quad U(1) \quad \text{by} \quad T^2, \quad (6a)$$

$$\Delta_1 \bar{\psi}^C T^1 \psi, \quad U(1) \quad \text{by} \quad T^3, \quad (6b)$$

$$\Delta_2 \bar{\psi}^C T^2 \psi, \quad SU(2) \quad \text{by} \quad T^1, T^2, T^3 \quad (6c)$$

$$\Delta_3 \bar{\psi}^C T^3 \psi, \quad U(1) \quad \text{by} \quad T^1, \quad (6d)$$

where $\psi^C \equiv C\bar{\psi}^T$ denotes the charge conjugate spinor, and C is a charge conjugation matrix, defined by $C^{-1}\gamma_\mu C = -\gamma_\mu^T$ and $C = -C^T$. In Eqs. (6a) – (6d) we also indicated the symmetry of the Majorana mass terms and the generators of the corresponding groups. Notice that the original $U(1)_V$ is always broken.

The parity properties of the Dirac and Majorana mass terms are summarized in Table I.

TABLE I. Parity properties of Dirac and Majorana mass terms.

	$P_1 = \gamma^1$	$P_2 = \gamma^0\gamma^2$	$P_3 = \gamma^5\gamma^1$	$P_4 = \gamma^3\gamma^1$
$\bar{\psi}\psi$	–	–	+	+
$\bar{\psi}T^1\psi$	–	–	–	–
$\bar{\psi}T^2\psi$	+	–	+	–
$\bar{\psi}T^3\psi$	+	–	–	+
$\bar{\psi}^C\psi$	+	–	+	+
$\bar{\psi}^CT^1\psi$	+	–	–	–
$\bar{\psi}^CT^2\psi$	–	–	+	–
$\bar{\psi}^CT^3\psi$	–	–	–	+

While the original field ψ belongs to the fundamental representation of the (pre-flavor) $SU(2)$ group, the charge conjugate field ψ^C belongs, in general, to the anti-fundamental representation. It is clear that the diquark fields such as the Majorana mass terms in Eq. (6), could belong either to the (antisymmetric) singlet representation (Δ_2), or to the (symmetric) triplet one (Δ , Δ_1 and Δ_3).

Since the most attractive diquark channel is antisymmetric (e.g., antitriplet in the case of $SU(3)$ color group), it becomes obvious that the color symmetry breaking order parameter would be $SU(2)$ singlet as that in Eq. (6c). Note that this order parameter is odd under the parity defined in Eq. (4), but even under the parity P_3 (see Table I).

III. HARD DENSE LOOP APPROXIMATION FOR THE GLUON PROPAGATOR

In dense media, the polarization effects play very important role. Below we derive the polarization tensor in the so-called hard dense loop (HDL) approximation. This approximation corresponds to dealing with semi-hard (external) gluon lines, $\alpha \ll k \ll \mu$, and hard (internal) fermion lines. Note that there are no one loop corrections with internal gluon and ghost lines in HDL approximation. Our notation and some formulas used in this section are given in Appendix A.

The analytical expression for the polarization tensor reads

$$\Pi^{\mu\nu AB}(k) = 4i\pi\alpha N_f \int \frac{dq_0 d^2q}{(2\pi)^3} \text{tr} [\gamma^\mu T^A G(q) \gamma^\nu T^B G(q+k)], \quad (7)$$

where α is a (dimensionful) coupling constant of $(2+1)$ -dimensional QCD and N_f counts the number of the four-component quark flavors, not the pre-flavors.

In derivation of the HDL polarization tensor, it is sufficient to consider only massless quarks. The corresponding propagator is given by

$$G^{(0)}(q) = i\gamma^0 \left[\frac{1}{q_0 + \epsilon_q^+ - i\varepsilon} \Lambda_q^{(+)} + \frac{1}{q_0 - \epsilon_q^- + i\varepsilon \text{sign}(\epsilon_q^-)} \Lambda_q^{(-)} \right]. \quad (8)$$

where $\epsilon_p^\pm = |\vec{p}| \pm \mu$ and the projectors $\Lambda_p^{(\pm)}$ are defined in Eq. (A12) in Appendix A. Note that ε is a small positive constant that prescribes the proper integration contour around the poles on the real q_0 -axis. At the end of calculation, this parameter should be taken to zero.

By substituting the quark propagator into Eq. (7), we arrive at

$$\begin{aligned} \Pi^{\mu\nu AB}(k) \simeq & -2i\pi\alpha N_f \delta^{AB} \int \frac{dq_0 d^2q}{(2\pi)^3} \left[\frac{\text{tr} [\gamma^\mu \gamma^0 \Lambda_q^{(-)} \gamma^\nu \gamma^0 \Lambda_{q+k}^{(-)}]}{(q_0 - \epsilon_q^- + i\varepsilon \text{sign}(\epsilon_q^-)) (q_0 + k_0 - \epsilon_{q+k}^- + i\varepsilon \text{sign}(\epsilon_{q+k}^-))} \right. \\ & + \frac{\text{tr} [\gamma^\mu \gamma^0 \Lambda_q^{(+)} \gamma^\nu \gamma^0 \Lambda_{q+k}^{(-)}]}{(q_0 + \epsilon_q^+ - i\varepsilon) (q_0 + k_0 - \epsilon_{q+k}^- + i\varepsilon \text{sign}(\epsilon_{q+k}^-))} + \frac{\text{tr} [\gamma^\mu \gamma^0 \Lambda_q^{(-)} \gamma^\nu \gamma^0 \Lambda_{q+k}^{(+)}]}{(q_0 - \epsilon_q^- + i\varepsilon \text{sign}(\epsilon_q^-)) (q_0 + k_0 + \epsilon_{q+k}^+ - i\varepsilon)} \left. \right]. \quad (9) \end{aligned}$$

Here we dropped one term in the integrand that contains its both poles in the upper half of the q_0 complex plane. Such a term does not give any contribution after the integration over q_0 is performed.

In the HDL approximation ($|\vec{k}| \ll |\vec{q}| \sim \mu$), the traces appearing in the last expression are simple (see Appendix A):

$$\text{tr} \left[\gamma^\mu \gamma^0 \Lambda_q^{(-)} \gamma^\nu \gamma^0 \Lambda_{q+k}^{(-)} \right] \simeq \text{tr} \left[\gamma^\mu \gamma^0 \Lambda_q^{(-)} \gamma^\nu \gamma^0 \Lambda_q^{(-)} \right] = 2 \left(g^{\mu 0} + \frac{\vec{q}^\mu}{|\vec{q}|} \right) \left(g^{\nu 0} + \frac{\vec{q}^\nu}{|\vec{q}|} \right), \quad (10)$$

$$\text{tr} \left[\gamma^\mu \gamma^0 \Lambda_q^{(+)} \gamma^\nu \gamma^0 \Lambda_{q+k}^{(-)} \right] \simeq \text{tr} \left[\gamma^\mu \gamma^0 \Lambda_q^{(+)} \gamma^\nu \gamma^0 \Lambda_q^{(-)} \right] = -2g^{\mu\nu} + 2g^{\mu 0} g^{\nu 0} - 2 \frac{\vec{q}^\mu}{|\vec{q}|} \frac{\vec{q}^\nu}{|\vec{q}|}, \quad (11)$$

$$\text{tr} \left[\gamma^\mu \gamma^0 \Lambda_q^{(-)} \gamma^\nu \gamma^0 \Lambda_{q+k}^{(+)} \right] \simeq \text{tr} \left[\gamma^\mu \gamma^0 \Lambda_q^{(-)} \gamma^\nu \gamma^0 \Lambda_q^{(+)} \right] = -2g^{\mu\nu} + 2g^{\mu 0} g^{\nu 0} - 2 \frac{\vec{q}^\mu}{|\vec{q}|} \frac{\vec{q}^\nu}{|\vec{q}|}. \quad (12)$$

In calculation of $\Pi^{\mu\nu}(k)$, we first perform the integration over q_0 using the residue theorem. The result reads (here we drop the overall δ^{AB})

$$\begin{aligned} \Pi^{\mu\nu}(k) \simeq & \frac{\alpha N_f}{\pi} \int d^2 q \left(g^{\mu 0} + \frac{\vec{q}^\mu}{|\vec{q}|} \right) \left(g^{\nu 0} + \frac{\vec{q}^\nu}{|\vec{q}|} \right) \left(\frac{\theta(\mu - |\vec{q}|) \theta(|\vec{q} + \vec{k}| - \mu)}{|\vec{q}| - |\vec{q} + \vec{k}| + k_0 + i\varepsilon} - \frac{\theta(|\vec{q}| - \mu) \theta(\mu - |\vec{q} + \vec{k}|)}{|\vec{q}| - |\vec{q} + \vec{k}| + k_0 - i\varepsilon} \right) \\ & + \frac{\alpha N_f}{\pi} \int d^2 q \left(g^{\mu\nu} - g^{\mu 0} g^{\nu 0} + \frac{\vec{q}^\mu}{|\vec{q}|} \frac{\vec{q}^\nu}{|\vec{q}|} \right) \left(\frac{\theta(|\vec{q}| - \mu)}{|\vec{q}| + |\vec{q} + \vec{k}| - k_0 - i\varepsilon} + \frac{\theta(|\vec{q} + \vec{k}| - \mu)}{|\vec{q}| + |\vec{q} + \vec{k}| + k_0 - i\varepsilon} \right). \end{aligned} \quad (13)$$

While the first term in this expression is finite, the second term contains a linear ultraviolet divergency. As usual, we drop the ultraviolet divergency from the polarization tensor in the HDL approximation (it would go away after the same renormalization procedure as in the case of zero chemical potential).

The leftover angular integration could be most easily done by going to a fixed coordinate framework [e.g., such that $\vec{k} = (|\vec{k}|, 0)$]. By doing so, we eventually arrive at the following result for the polarization tensor in the HDL approximation:

$$\Pi^{00}(k_0, \vec{k}) = \Pi_l(k_0, \vec{k}), \quad (14)$$

$$\Pi^{0i}(k_0, \vec{k}) = k_0 \frac{k^i}{|\vec{k}|^2} \Pi_l(k_0, \vec{k}), \quad (15)$$

$$\Pi^{ij}(k_0, \vec{k}) = \left(\delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2} \right) \Pi_t(k_0, \vec{k}) + \frac{k^i k^j}{|\vec{k}|^2} \frac{k_0^2}{|\vec{k}|^2} \Pi_l(k_0, \vec{k}), \quad (16)$$

where

$$\Pi_l(k_0, \vec{k}) = \frac{2N_f \alpha \mu}{\pi} \int_0^{\pi/2} d\varphi \left(\frac{|\vec{k}| \cos \varphi}{k_0 - |\vec{k}| \cos \varphi + i\varepsilon} - \frac{|\vec{k}| \cos \varphi}{k_0 + |\vec{k}| \cos \varphi - i\varepsilon} \right) = M^2 \left(\theta(k^2) \sqrt{\frac{k_0^2}{k^2}} - 1 - i\theta(-k^2) \sqrt{\frac{k_0^2}{-k^2}} \right), \quad (17)$$

$$\Pi_t(k_0, \vec{k}) = \frac{2N_f \alpha \mu}{\pi} \int_0^{\pi/2} d\varphi \left(\frac{|\vec{k}| \cos \varphi \sin^2 \varphi}{k_0 - |\vec{k}| \cos \varphi + i\varepsilon} - \frac{|\vec{k}| \cos \varphi \sin^2 \varphi}{k_0 + |\vec{k}| \cos \varphi - i\varepsilon} + 2 \cos^2 \varphi \right) = M^2 - \frac{k^2}{|\vec{k}|^2} \Pi_l(k_0, \vec{k}) \quad (18)$$

with $k^2 = k_0^2 - |\vec{k}|^2$. Here we use the notation $M^2 = 2\alpha\mu N_f$ for the Debye mass. We should point out that the expression for $\Pi_l(k_0, \vec{k})$, as written in Eq. (17), is valid only for real values of k_0 . The expression for $\Pi_l(k_0, \vec{k})$ as a function of complex k_0 has the following form:

$$\Pi_l(k_0, \vec{k}) = M^2 \left(\frac{1}{\sqrt{1 - (|\vec{k}|/k_0)^2}} - 1 \right). \quad (19)$$

Our expressions in Eqs. (17) and (18) resemble the results in the $(3+1)$ -dimensional case [37–39]. The corresponding polarization tensor has a nonzero imaginary part for space-like gluon momenta. This is related to the so-called Landau damping of the gluon field.

The gluon polarization tensor is transverse

$$k^\mu \Pi_{\mu\nu}(k_0, \vec{k}) = 0, \quad (20)$$

and the following representation holds:

$$\Pi_{\mu\nu} = -O_{\mu\nu}^{(1)} \Pi_t + O_{\mu\nu}^{(2)} (\Pi_t - M^2). \quad (21)$$

Here we make use of the projectors of the magnetic, electric and longitudinal gluon modes introduced in Ref. [7],

$$O^{(1)} = g^{\mu\nu} - u^\mu u^\nu + \frac{\vec{k}^\mu \vec{k}^\nu}{|\vec{k}|^2}, \quad (22a)$$

$$O^{(2)} = u^\mu u^\nu - \frac{\vec{k}^\mu \vec{k}^\nu}{|\vec{k}|^2} - \frac{k^\mu k^\nu}{k^2}, \quad (22b)$$

$$O^{(3)} = \frac{k^\mu k^\nu}{k^2}, \quad (22c)$$

with $u_\mu = (1, 0, 0, 0)$ and $\vec{k}_\mu = k_\mu - (u \cdot k) u_\mu$.

Then, we rewrite the inverse gluon propagator as

$$\left(\mathcal{D}^{AB}(k_0, \vec{k}) \right)_{\mu\nu}^{-1} = i\delta^{AB} (k^2 - \Pi_t) O_{\mu\nu}^{(1)} + i\delta^{AB} (k^2 + \Pi_t - M^2) O_{\mu\nu}^{(2)} + i\delta^{AB} \frac{k^2}{\xi} O_{\mu\nu}^{(3)}, \quad (23)$$

where ξ is a gauge parameter. Then, by making use of the properties of the projection operators, we invert this expression and arrive at the final form of the gluon propagator in HDL approximation

$$\mathcal{D}_{\mu\nu}^{AB}(k_0, \vec{k}) = -i\delta^{AB} \frac{1}{k^2 - \Pi_t} O_{\mu\nu}^{(1)} - i\delta^{AB} \frac{1}{k^2 + \Pi_t - M^2} O_{\mu\nu}^{(2)} - i\delta^{AB} \frac{\xi}{k^2} O_{\mu\nu}^{(3)}. \quad (24)$$

By using the conventional assumption that the pairing dynamics is dominated by the momenta $|k_0| \ll |\vec{k}|$, we arrive at the following approximate expression of the propagator in Euclidean space ($k_0 = ik_4$):

$$i\mathcal{D}_{\mu\nu}^{AB}(ik_4, \vec{k}) \simeq -\delta^{AB} \frac{|\vec{k}|}{|\vec{k}|^3 + M^2|k_4|} O_{\mu\nu}^{(1)} - \delta^{AB} \frac{1}{k_4^2 + |\vec{k}|^2 + M^2} O_{\mu\nu}^{(2)} - \delta^{AB} \frac{\xi}{k_4^2 + |\vec{k}|^2} O_{\mu\nu}^{(3)}. \quad (25)$$

IV. SCHWINGER-DYSON EQUATION

The mechanism of breaking of color symmetry in dense QCD is well known. The dynamics of quasiparticles around the Fermi surface is affected by the famous Cooper instability. This causes the perturbative vacuum to rearrange so that an energy gap is formed in the spectrum of quasiparticles. This also is accompanied by the appearance of a color antitriplet (in the case of three colors) condensate that “breaks” the original gauge symmetry $SU(3)_c$ down to $SU(2)_c$ subgroup. As we discussed in the preceding section, the corresponding order parameter leaves the global $SU(2)$ pre-flavor symmetry intact.

In passing we note that because of the Higgs mechanism, five out of the original eight gluons become massive. The value of the corresponding mass functions at zero momentum is expected to be of order $\alpha\mu$. This might suggest that the mentioned five gluons are rather inefficient in providing interaction in the diquark pairing. If so, one should take this into account when studying the SD equation. However, we assume that the Meissner mass function of gluons quickly vanishes (approaching the HDL approximation) as we go to the region of momenta larger than the superconducting gap. This is what happens in the $(3+1)$ -dimensional model [7], and we assume that this should be still valid in $(2+1)$ dimensions.

Our strategy is to use the HDL improved rainbow approximation for the SD equation and then to check whether it is reliable [as it happens in the $(3+1)$ -dimensional case] for an asymptotically large chemical potential. The SD equation written in this approximation is:

$$(G(p))^{-1} = \left(G^{(0)}(p) \right)^{-1} + 4\pi\alpha \int \frac{dq_0 d^2q}{(2\pi)^3} \sum_A \gamma^\mu \begin{pmatrix} T^A & 0 \\ 0 & -(T^A)^T \end{pmatrix} G(q) \gamma^\nu \begin{pmatrix} T^A & 0 \\ 0 & -(T^A)^T \end{pmatrix} \mathcal{D}_{\mu\nu}(q-p), \quad (26)$$

where $\mathcal{D}_{\mu\nu}(k)$ is the propagator of gluons with the overall factor δ^{AB} omitted. We denote the full quark propagator by $G(p)$, and the perturbative one by $G^{(0)}(p)$.

As is argued in Sec. II, the order parameter should be a color anti-triplet and a pre-flavor singlet, $\varepsilon^{3ab}\langle\bar{\psi}_a^C T^2\psi_b\rangle$. Therefore, the inverse of the full quark propagator should have the following general structure (neglecting the wave function renormalization):

$$(G(p))^{-1} = -i \begin{pmatrix} \not{p} + \mu\gamma^0 & \Delta_p \\ \tilde{\Delta}_p & \not{p} - \mu\gamma^0 \end{pmatrix}, \quad (27)$$

where $\tilde{\Delta}_p = \gamma^0 \Delta_p^\dagger \gamma^0$ and $(\Delta_p)_{ab} \equiv \varepsilon_{ab3} T^2 (\Delta_p^- \Lambda_p^- + \Delta_p^+ \Lambda_p^+)$.

By inverting the expression in Eq. (27), we arrive at the following propagator:

$$G(p) = i \begin{pmatrix} R_1 & \Sigma \\ \tilde{\Sigma} & R_2 \end{pmatrix}, \quad (28)$$

where

$$R_1 = \mathcal{I}_1 \gamma^0 \left(\frac{(p_0 + \epsilon_p^-) \Lambda_p^-}{p_0^2 - (\epsilon_p^-)^2 - |\Delta_p^-|^2} + \frac{(p_0 - \epsilon_p^+) \Lambda_p^+}{p_0^2 - (\epsilon_p^+)^2 - |\Delta_p^+|^2} \right) + \mathcal{I}_2 \gamma^0 \left(\frac{\Lambda_p^-}{p_0 - \epsilon_p^-} + \frac{\Lambda_p^+}{p_0 + \epsilon_p^+} \right), \quad (29)$$

$$R_2 = \mathcal{I}_1 \gamma^0 \left(\frac{(p_0 - \epsilon_p^-) \Lambda_p^+}{p_0^2 - (\epsilon_p^-)^2 - |\Delta_p^-|^2} + \frac{(p_0 + \epsilon_p^+) \Lambda_p^-}{p_0^2 - (\epsilon_p^+)^2 - |\Delta_p^+|^2} \right) + \mathcal{I}_2 \gamma^0 \left(\frac{\Lambda_p^+}{p_0 + \epsilon_p^-} + \frac{\Lambda_p^-}{p_0 - \epsilon_p^+} \right), \quad (30)$$

$$\Sigma_{ab} = \varepsilon_{ab3} T^2 \left(\frac{\Delta_p^- \Lambda_p^+}{p_0^2 - (\epsilon_p^-)^2 - |\Delta_p^-|^2} + \frac{\Delta_p^+ \Lambda_p^-}{p_0^2 - (\epsilon_p^+)^2 - |\Delta_p^+|^2} \right), \quad (31)$$

and $(\mathcal{I}_1)_{ab} = \delta_{ab} - \delta_a^3 \delta_b^3$, $(\mathcal{I}_2)_{ab} = \delta_a^3 \delta_b^3$.

Thus, the gap equation reads

$$\Delta_p^- \Lambda_p^- + \Delta_p^+ \Lambda_p^+ = -i \frac{8\pi\alpha}{3} \int \frac{dq_4 d^2 q}{(2\pi)^3} \left(\frac{\gamma^\mu \Delta_q^- \Lambda_q^+ \gamma^\nu}{q_4^2 + (\epsilon_q^-)^2 + |\Delta_q^-|^2} + \frac{\gamma^\mu \Delta_q^+ \Lambda_q^- \gamma^\nu}{q_4^2 + (\epsilon_q^+)^2 + |\Delta_q^+|^2} \right) \mathcal{D}_{\mu\nu}(iq_4 - ip_4, \vec{q} - \vec{p}). \quad (32)$$

Among two gap functions, Δ_p^- and Δ_p^+ , only the first one plays the role of the gap in the quark spectrum around the Fermi surface. In addition, the main contribution in the right hand side of Eq. (32) comes from the first term also proportional to Δ_p^- . As a result, the equation for Δ_p^- approximately decouples,

$$\Delta_p^- \simeq -i \frac{4\pi\alpha}{3} \int \frac{dq_4 d^2 q}{(2\pi)^3} \frac{\Delta_q^- \text{tr} [\gamma^\mu \Lambda_q^+ \gamma^\nu \Lambda_p^-]}{q_4^2 + (\epsilon_q^-)^2 + |\Delta_q^-|^2} \mathcal{D}_{\mu\nu}(iq_4 - ip_4, \vec{q} - \vec{p}). \quad (33)$$

By taking the trace and performing the angular integration (see Appendix A for details), we arrive at the following approximate equation:

$$\Delta_p^- \simeq \frac{\alpha}{6\pi} \int \frac{dq_4 dq \Delta_q^-}{q_4^2 + (\epsilon_q^-)^2 + |\Delta_q^-|^2} \left[\frac{1}{\sqrt{(q-p)^2 + (M^2|q_4 - p_4|)^{2/3}}} + \frac{2}{\sqrt{(q-p)^2 + (q_4 - p_4)^2 + M^2}} + \frac{\xi}{\sqrt{(q-p)^2 + (q_4 - p_4)^2}} \right]. \quad (34)$$

To get a analytical estimate for the solution, we assume that the gap is only a function of the time component of the momentum, i.e., $\Delta^-(p_4, p) \simeq \Delta^-(p_4, \mu)$. Then, we can easily perform the integration over $\epsilon_q^- = q - \mu$. The approximate result is

$$\Delta^-(p_4) \simeq \frac{\alpha}{6} \int \frac{dq_4 \Delta^-(q_4)}{\sqrt{q_4^2 + |\Delta_q^-|^2}} \left[\frac{1}{(M^2|q_4 - p_4|)^{1/3}} + \frac{2}{M} + \frac{\xi}{|q_4 - p_4|} \right]. \quad (35)$$

Because of the approximations, this last equation makes sense only with a finite ultraviolet cutoff which should be chosen at the scale of M .

Now by making use of the same method as in Ref. [7] and using the Landau gauge ($\xi = 0$), we straightforwardly obtain an analytical solution (see Appendix B for details):

$$\Delta_0^-(p_4) = c|\Delta_0^-| \left(\frac{|\Delta_0^-|}{p_4} \right)^{1/6} J_1 \left(\sqrt{\frac{4\alpha}{(M^2 p_4)^{1/3}}} \right), \quad c \approx 1.6, \quad (36)$$

with the overall normalization constant c chosen so that $\Delta_0^-(p_4)|_{p_4 \approx |\Delta_0^-|} = |\Delta_0^-|$. The value of the gap itself is equal to

$$|\Delta_0^-| \simeq b \frac{\alpha^3}{M^2} = b \frac{\alpha^2}{2\mu}, \quad (37)$$

where $b \approx 0.331$. Note that we neglected the effect of electric gluon modes because their contribution is strongly suppressed compared to the effect of the magnetic modes.

How reliable is the HDL improved rainbow approximation in $(2+1)$ -dimensional QCD? Because of a singular nature of the gauge fixing term in Eq. (35), we see that the above result for the gap is subject to a large gauge dependent correction in a general covariant gauge. This indicates that the HDL improved rainbow approximation may not be reliable. In the next section, we will show that the origin of this problem in $(2+1)$ -dimensional QCD is the phenomenon of nondecoupling of the dynamics of the fermion pairing from the nonperturbative infrared dynamics. This phenomenon is intrinsic for lower dimensional field theories and does not take place in dense QCD in $(3+1)$ dimensions. We will also discuss a way of solving this problem.

V. NONDECOUPLING PHENOMENA IN THE GAP DYNAMICS IN $(2+1)$ -DIMENSIONAL QCD

Let us first recall that the HDL improved rainbow approximation is reliable in $(3+1)$ -dimensional dense QCD at asymptotically high quark densities. This is related to the fact that a large chemical potential μ provides a large value of the gap in $(3+1)$ dimensions, namely, $|\Delta_0^-| \gg \Lambda_{QCD}$ for a sufficiently large μ [6–9]. This is enough to justify the use of the one gluon exchange approximation for the kernel of the SD equation. Indeed, the integration over the gluon momentum in the SD equation is dynamically cut at a momentum $k \simeq |\Delta_0^-| \gg \Lambda_{QCD}$ in infrared, insuring that the dominant contribution is given by the hard gluons only. As a result, the diquark pairing dynamics is completely decoupled from the strong infrared dynamics.

The situation in dense QCD in $(2+1)$ dimensions is different. While the contribution of hard fermion loops still dominates in the polarization operator for large μ , the value of the gap (37) obtained in the HDL improved rainbow approximation in the Landau gauge is small. Therefore, a large region of soft gluons contributes to the SD equation (moreover, it dominates). It is clear therefore that the one gluon exchange approximation is not reliable in this case.

What is the physical origin of such a drastic difference between the $(3+1)$ - and $(2+1)$ -dimensional cases? We believe it is a diminishing role of the Fermi surface in the fermion pairing dynamics in lower dimensions. Indeed, the main effect of the Fermi surface in the fermion pairing is reducing the initial dimension $d+1$ to the effective dimension $1+1$. This is crucial for $d=3$. However, for example, for $d=1$ there is no such a reduction at all: there is $1+1$ dimensional dynamics from the outset!¹ The present case with $d=2$ is intermediate, and our analysis shows how different it is from the $d=3$ case.

The importance of the region of soft gluon momenta in the SD equation implies that the pairing dynamics in $(2+1)$ -dimensional QCD is intimately connected with the strong infrared dynamics. In fact, there is no decoupling of these two dynamics even for asymptotically large values of the chemical potential.

Let us show that this point is directly related to the existence of an upper bound for the value of the gap in the $(2+1)$ -dimensional QCD. In particular, $|\Delta_0^-|$ cannot be much larger than α . Indeed, if the value of the gap were much larger than α , then the HDL improved rainbow approximation would be reliable. But the only solution of the corresponding equation is given in Eq. (37), showing that $|\Delta_0^-| \ll \alpha$. Thus, the assumption of large $|\Delta_0^-|$ is self-contradictory.

Therefore in order to solve the problem for large μ in $(2+1)$ dimension, one should put the non-perturbative infrared dynamics under control. Does it imply that expression (37) obtained in the HDL improved rainbow approximation is irrelevant? We do not think so: we will argue that it can be qualitatively correct in the case of a large number of fermion flavors N_f .

¹In passing, we note that in the $1+1$ case, in the HDL approximation, the Debye screening mass is proportional to α_2 (where α_2 is the gauge coupling) and is independent of μ at all! The Debye mass actually coincides with the famous Schwinger mass, appearing for $\mu=0$ in $1+1$ [see also a discussion in Ref. [40]]. This observation certainly implies the breakdown of the HDL expansion in $1+1$ dimensional QCD.

The fact that the $1/N_f$ expansion can be useful in $(2+1)$ -dimensional QED and QCD was recognized long ago [33–36]. The point is that because these models are renormalizable theories, they are asymptotically free for any number N_f . Though being very nontrivial, the $1/N_f$ expansion is helpful in putting under control the infrared dynamics. The crucial point is the selection of a “right” gauge in the leading order in $1/N_f$. In particular, appropriate Ward identities have to be satisfied in that gauge. In other gauges, the result can be found by gauge-transforming Green functions from the “right” gauge to those gauges. The description of this very nontrivial procedure can be found in Ref. [41]. For our purposes, it is enough to know that though the “right” gauge is different from the Landau one, the results obtained in the improved rainbow approximation in the latter are qualitatively reliable: its special role amongst other covariant gauges is connected with the approximate validity of the Ward identities in the improved rainbow approximation using the bare vertices.

It is reasonable to assume that the Landau gauge is special in our problem either. Indeed, let us recall that, because of the Miessner effect, the quark gluon vertices should necessarily receive non-perturbative pole corrections [42]. This is a rather general and a model independent property that is related to the Ward identities [43,44] for the vortex function. Following Ref. [42], one can show that in a general covariant gauge the contribution of those pole terms in the SD equation is large, thus destroying the validity of the approximation with bare vertices. The solution to a similar problem was suggested long time ago by Cornwall and Norton [44]. The idea is to choose the gauge where the approximation of the SD equation with bare vertices is self-consistent. In their model, it was the Landau gauge. With this gauge choice, it was possible to show that potentially dangerous contributions cancel from the SD equation. This is possible due to the fact that the pole contribution to the vertex is pure longitudinal. The Landau gauge is special because the gluon propagator is completely transverse. Thus, when used in the SD equation, such gluon propagator annihilates the dangerous infrared poles.

Using the same arguments, we see that the Landau gauge is the best gauge among the general covariant gauges in the problem at hand too. We should note, however, that our situation differs from that in Ref. [44]. The Lorentz symmetry is broken in dense QCD by a nonzero chemical potential. As a result, the pole structure of the quark gluon vertex is proportional to $\tilde{p}^\mu - \tilde{q}^\mu = (p^0 - q^0, (\vec{p} - \vec{q})/3)$, rather than $p^\mu - q^\mu$ as in Ref. [44]. While the four-vector $\tilde{p}^\mu - \tilde{q}^\mu$ is indeed annihilated by the magnetic modes of gluons, it is only partially annihilated by the electric ones. It is fortunate that the electric gluon modes are screened much stronger and their effect on diquark pairing is negligible in the model at hand. So, we could still justify the use of the Landau gauge, in which all most dangerous infrared contributions are canceled from the SD equation.

These arguments lead us to believe that expression (37) can be qualitatively reliable for large N_f . We rewrite the result in the following form:

$$|\Delta_0^-| \simeq \bar{b} \frac{\bar{\alpha}^2}{2N_f^2 \mu}, \quad (38)$$

where $\bar{\alpha} = N_f \alpha$ and the constant \bar{b} is of order one. We got this estimate by making use of conventional large N_f rescaling of the gauge coupling. Though a color-flavor structure of the condensate can vary with N_f , we expect that the magnitude of the gap should be similar for different, but same order, large N_f .

VI. CONCLUSION

We studied the dynamical generation of the color superconducting order parameter in $(2+1)$ -dimensional QCD at asymptotically large baryon density and zero temperature. For this purpose, we used the conventional method of the Schwinger-Dyson equation in the HDL improved rainbow approximation. It was found that the order parameter in this theory is similar to that in the $S2C$ phase of the $(3+1)$ -dimensional two flavor QCD.

By solving the SD equation analytically, we obtained an approximate solution for the color superconducting gap. The parametric dependence of the result on the coupling constant is given by a power law. This is qualitatively different from the situation in the $(3+1)$ -dimensional QCD, where the expression for the gap is exponentially small for small values of the coupling constant [6–9].

The analysis of the solution of the SD equation revealed a qualitatively new [with respect to the $(3+1)$ -dimensional case] phenomenon: nondecoupling of the dynamics of the fermion pairing from the nonperturbative infrared one, even in the case of asymptotically large values of the chemical potential. We believe that this phenomenon is common in lower dimensional models and deserves further study.

It would be also interesting to study the interplay between the dynamics of color and chiral condensates in $(2+1)$ -dimensional QCD. It is known that, at zero quark density, a chiral condensate can occur only if the number of quark flavors is less than a critical value N_f^{cr} , which is a function of the number of colors N_c [35,36]. An interesting question

is whether, in this case, there is a phase transition which restores chiral symmetry at a critical value of the chemical potential and how the character of this phase transition depends on the values of N_f and N_c .

In the analysis of the SD equation we assumed that the Meissner effect is mostly irrelevant for the pairing dynamics in dense quark matter. While this was well justified in $(3+1)$ -dimensional QCD at asymptotic densities [7,8,45], it remains an open issue in the $(2+1)$ -dimensional case. Indeed, our present analysis suggests that the dominant contribution to the gap equation comes from the region of momenta not much larger than $|\Delta^-|$. The Meissner effect, therefore, could play a significant role in modifying the gluon interaction in the diquark channel. It would be interesting to study this problem in more detail.

It is interesting to point out that, because of close similarity of the model studied here with that of the $(3+1)$ -dimensional two flavor QCD, one should also expect the appearance of five light ($M_{dq} \ll |\Delta_0|$) pseudo-Nambu-Goldstone diquarks in the low energy spectrum [42]. The argument for the existence of such states is essentially the same as in the $(3+1)$ -dimensional case, see Ref. [42]. The only difference should appear in the hierarchy of the scales.

At last, we would like to mention that some relativistic $(2+1)$ -dimensional gauge models are used by some theorists for describing high temperature superconductivity in cuprates [46]. It would be interesting to see whether the color superconductivity could anyhow be related to the ordinary superconductivity in planar systems.

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NOTE

While writing our paper, we learned that a partially overlapping study is being done by Prashanth Jaikumar and Ismail Zahed.

APPENDIX A: NOTATION AND USEFUL FORMULAS

We use the four-dimensional representation of the Dirac γ -matrices. In addition to the three matrices γ^μ ($\mu = 0, 1, 2$) there are two more matrices that anticommute with γ^μ as well as between themselves, γ^3 and γ^5 [here we use the same notation for the γ -matrices as one would use in the $(3+1)$ -dimensional case].

While studying the color superconductivity in the model, it is convenient to introduce the charge conjugate spinors as follows:

$$\psi^C(x) = C\bar{\psi}^T(x), \quad \bar{\psi}(x) = -[\psi^C(x)]^T C^\dagger, \quad (\text{A1})$$

$$\bar{\psi}^C(x) = -\psi^T(x)C^\dagger, \quad \psi(x) = C[\bar{\psi}^C(x)]^T, \quad (\text{A2})$$

where C is a unitary charge conjugation matrix, defined by $C^{-1}\gamma_\mu C = -\gamma_\mu^T$ and $C = -C^T$.

Note that in momentum space,

$$\psi^C(p) = C\bar{\psi}^T(-p), \quad \bar{\psi}(p) = -[\psi^C(-p)]^T C^\dagger, \quad (\text{A3})$$

$$\bar{\psi}^C(p) = -\psi^T(-p)C^\dagger, \quad \psi(p) = C[\bar{\psi}^C(-p)]^T. \quad (\text{A4})$$

The Lagrangian density of QCD could be written as

$$\mathcal{L}_{QCD} = \frac{1}{2}\bar{\psi}(\not{p} + \mu\gamma^0)\psi + \frac{1}{2}\bar{\psi}^{Ci}(\not{p} - \mu\gamma^0)\psi^C + \frac{1}{2}\bar{\psi}\gamma^\mu\hat{A}_\mu\psi^i - \frac{1}{2}\bar{\psi}^C\gamma^\mu\hat{A}_\mu^T\psi^C - \frac{1}{16\pi\alpha}\text{Tr}(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}). \quad (\text{A5})$$

The eight component Majorana spinor is defined by

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi \\ \psi^C \end{pmatrix}. \quad (\text{A6})$$

In this new notation, the bare fermion propagator reads

$$\left(G^{(0)}(p)\right)^{-1} = -i \begin{pmatrix} \not{p} + \mu\gamma^0 & 0 \\ 0 & \not{p} - \mu\gamma^0 \end{pmatrix}. \quad (\text{A7})$$

Note the following convenient representations for the diagonal elements of this propagator:

$$\not{p} + \mu\gamma^0 = \gamma^0 \left[(p_0 - \epsilon_p^-) \Lambda_p^{(+)} + (p_0 + \epsilon_p^+) \Lambda_p^{(-)} \right], \quad (\text{A8})$$

$$\not{p} - \mu\gamma^0 = \gamma^0 \left[(p_0 - \epsilon_p^+) \Lambda_p^{(+)} + (p_0 + \epsilon_p^-) \Lambda_p^{(-)} \right], \quad (\text{A9})$$

$$(\not{p} + \mu\gamma^0)^{-1} = \gamma^0 \left[\frac{1}{p_0 + \epsilon_p^+} \Lambda_p^{(+)} + \frac{1}{p_0 - \epsilon_p^-} \Lambda_p^{(-)} \right], \quad (\text{A10})$$

$$(\not{p} - \mu\gamma^0)^{-1} = \gamma^0 \left[\frac{1}{p_0 + \epsilon_p^-} \Lambda_p^{(+)} + \frac{1}{p_0 - \epsilon_p^+} \Lambda_p^{(-)} \right], \quad (\text{A11})$$

where $\epsilon_p^\pm = |\vec{p}| \pm \mu$ and the “on-shell” projectors are

$$\Lambda_p^{(\pm)} = \frac{1}{2} \left(1 \pm \frac{\vec{\alpha} \cdot \vec{p}}{|\vec{p}|} \right), \quad \vec{\alpha} \equiv \gamma^0 \vec{\gamma}. \quad (\text{A12})$$

The bare quark-quark-gluon vertex, in the eight component notation, is also a matrix,

$$\gamma^\mu \begin{pmatrix} T^A & 0 \\ 0 & -(T^A)^T \end{pmatrix}. \quad (\text{A13})$$

While working with the SD equation in Sec. IV, we encounter a few Dirac traces. Here are the results for most general ones:

$$\text{tr} \left[\gamma^\mu \Lambda_p^{(e)} \gamma^\nu \Lambda_q^{(e')} \right] = g^{\mu\nu} (1 + ee't) - 2ee'g^{\mu 0}g^{\nu 0}t + ee' \frac{\vec{q}^\mu \vec{p}^\nu + \vec{q}^\nu \vec{p}^\mu}{|\vec{q}||\vec{p}|} + \dots, \quad (\text{A14})$$

$$\begin{aligned} \text{tr} \left[\gamma^\mu \gamma^0 \Lambda_p^{(e)} \gamma^\nu \gamma^0 \Lambda_q^{(e')} \right] &= -g^{\mu\nu} (1 - ee't) + \left(g^{\mu 0} - e' \frac{\vec{q}^\mu}{|\vec{q}|} \right) \left(g^{\nu 0} - e \frac{\vec{p}^\nu}{|\vec{p}|} \right) \\ &\quad + \left(g^{\mu 0} - e \frac{\vec{p}^\mu}{|\vec{p}|} \right) \left(g^{\nu 0} - e' \frac{\vec{q}^\nu}{|\vec{q}|} \right) + \dots, \end{aligned} \quad (\text{A15})$$

where $e, e' = \pm 1$, $t = \cos \varphi$ is the cosine of the angle between two-vectors \vec{q} and \vec{p} , and irrelevant antisymmetric terms are denoted by ellipsis.

By making use of the traces above, we calculate the following expressions which appear in the SD equation:

$$O_{\mu\nu}^{(1)} \text{tr} \left[\gamma^\mu \Lambda_q^{(+)} \gamma^\nu \Lambda_p^{(-)} \right] = (1 - t) \frac{(q + p)^2}{q^2 + p^2 - 2qpt}, \quad (\text{A16a})$$

$$O_{\mu\nu}^{(2)} \text{tr} \left[\gamma^\mu \Lambda_q^{(+)} \gamma^\nu \Lambda_p^{(-)} \right] = (1 + t) \frac{(q - p)^2}{q^2 + p^2 - 2qpt} + (1 - t^2) \frac{2qp}{q^2 + p^2 - 2qpt + (q_4 - p_4)^2}, \quad (\text{A16b})$$

$$O_{\mu\nu}^{(3)} \text{tr} \left[\gamma^\mu \Lambda_q^{(+)} \gamma^\nu \Lambda_p^{(-)} \right] = (1 + t) \frac{(q - p)^2 + (q_4 - p_4)^2}{q^2 + p^2 - 2qpt + (q_4 - p_4)^2}, \quad (\text{A16c})$$

where $q \equiv |\vec{q}|$, $p \equiv |\vec{p}|$, $q_4 \equiv -iq_0$ and $p_4 \equiv -ip_0$. Making use of these expression, we perform the angular integration that appears in the SD equation,

$$\begin{aligned} q \int d\varphi \mathcal{D}_{\mu\nu}(q - p) \text{tr} \left[\gamma^\mu \Lambda_p^{(-)} \gamma^\nu \Lambda_q^{(+)} \right] &\approx i\pi \left[\frac{1}{\sqrt{(q - p)^2 + (M^2\omega)^{2/3}}} + \frac{2}{\sqrt{(q - p)^2 + \omega^2 + M^2}} \right. \\ &\quad \left. + \frac{\xi}{\sqrt{(q - p)^2 + \omega^2}} \right]. \end{aligned} \quad (\text{A17})$$

Here $M^2 = 2\alpha\mu N_f$ and $\omega = |q_4 - p_4|$. Although we performed the angular integration exactly, we dropped all subleading terms on the right hand of Eq. (A17), assuming that both q and p remain in the vicinity of the Fermi surface.

APPENDIX B: APPROXIMATE SOLUTION OF THE GAP EQUATION

In this Appendix we present an approximate analytical solution to the gap equation (35). In order to reduce the integral gap equation to a differential one, we introduce an infrared cutoff in the integral at the scale of $|\Delta_0^-|$, and we approximate the kernel in the integrand by its asymptotes:

$$\frac{1}{\sqrt{q_4^2 + |\Delta_q^-|^2}} \frac{1}{(M^2|q_4 - p_4|)^{1/3}} = \begin{cases} \frac{1}{(M^2|p_4|)^{1/3}q_4}, & \text{for } q_4 \leq p_4, \\ \frac{1}{M^{2/3}|q_4|^{4/3}}, & \text{for } q_4 \geq p_4. \end{cases} \quad (\text{B1})$$

We then arrive at the following equation:

$$\Delta^-(p_4) \simeq \frac{\nu^{1/3}}{3|p_4|^{1/3}} \int_{|\Delta_0^-|}^{p_4} \frac{dq_4}{q_4} \Delta^-(q_4) + \frac{\nu^{1/3}}{3} \int_{p_4}^{\infty} \frac{dq_4}{|q_4|^{4/3}} \Delta^-(q_4), \quad (\text{B2})$$

where

$$\nu \equiv \frac{\alpha^3}{M^2}. \quad (\text{B3})$$

It is straightforward to check that the integral equation is equivalent to the following second order differential equation

$$\frac{d^2 \Delta^-(p_4)}{dp_4^2} + \frac{4}{3p_4} \frac{d\Delta^-(p_4)}{dp_4} + \frac{\nu^{1/3}}{9p_4^{7/3}} \Delta^-(p_4) = 0, \quad (\text{B4})$$

along with the infrared and ultraviolet boundary conditions:

$$\left. \frac{d\Delta^-(p_4)}{dp_4} \right|_{p_4=|\Delta_0^-|} = 0, \quad (\text{B5})$$

$$\left(3p_4 \frac{d\Delta^-(p_4)}{dp_4} + \Delta^-(p_4) \right) \Big|_{p_4=\infty} = 0. \quad (\text{B6})$$

The general solution to the differential equation (B4) is given in terms of Bessel functions,

$$\Delta^-(p_4) = |\Delta_0^-| \left[C_1 \left(\frac{|\Delta_0^-|}{p_4} \right)^{1/6} J_1 \left(2 \frac{\nu^{1/6}}{p_4^{1/6}} \right) + C_2 \left(\frac{|\Delta_0^-|}{p_4} \right)^{1/6} Y_1 \left(2 \frac{\nu^{1/6}}{p_4^{1/6}} \right) \right]. \quad (\text{B7})$$

In order to satisfy the ultraviolet boundary condition in Eq. (B6), we must choose $C_2 = 0$. In passing we note, however, that when a finite ultraviolet cutoff is used in the gap equation (B2) (one might choose, for example, $q_4 = M$), the integration constant C_2 will be nonzero,

$$C_2 \simeq \frac{\pi}{2} \left(\frac{\nu}{M} \right)^{2/3} C_1. \quad (\text{B8})$$

As is easy to check, the effect of this nonzero constant is negligible in the leading order approximation.

Now, by satisfying the infrared boundary condition, we arrive at the following equation:

$$J_1 \left(2 \frac{\nu^{1/6}}{|\Delta_0^-|^{1/6}} \right) = \left(\frac{\nu}{|\Delta_0^-|} \right)^{1/6} J_2 \left(2 \frac{\nu^{1/6}}{|\Delta_0^-|^{1/6}} \right) - \left(\frac{\nu}{|\Delta_0^-|} \right)^{1/6} J_0 \left(2 \frac{\nu^{1/6}}{|\Delta_0^-|^{1/6}} \right). \quad (\text{B9})$$

This equation has infinitely many solutions, which correspond to different solutions to the gap equation. We must choose a single solution which gives the gap function without nodes. As one could check, this is also equivalent to choosing the solution with the largest value of the gap. Thus, we obtain $|\Delta_0^-| = b\nu$ where $b \approx 0.331$.

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